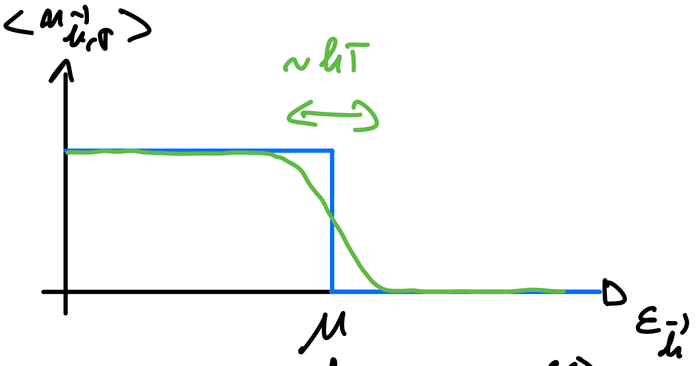
7. Fermi-Dirac Statistics

$$\langle n_{\vec{k},\sigma} \rangle = \frac{1}{e^{\beta[\varepsilon_{\vec{k}} - \mu]} + 1}$$

$$\frac{1}{1+e^{x}} \begin{cases} \to 0 & \text{as } x \to +\infty \\ \to 1 & \text{as } x \to -\infty \end{cases}$$

Statistics at low temperature

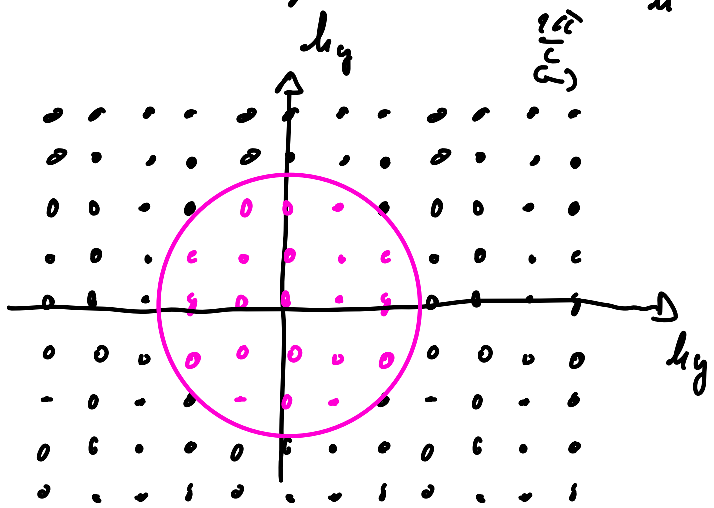


At $t=0$, all levels are full up to $\varepsilon_{\vec{k}} = \mu$ and empty above μ. $\varepsilon_F = \mu$ is then called the Fermi energy.

The occupied levels are called the Fermi sea. They satisfy

$$\varepsilon_k < \varepsilon_F \Leftrightarrow \frac{\hbar^2 k^2}{2m} < \varepsilon_F$$

$$\Rightarrow k_F = \sqrt{\frac{2m\mu}{\hbar^2}}$$ is the Fermi wavenumber

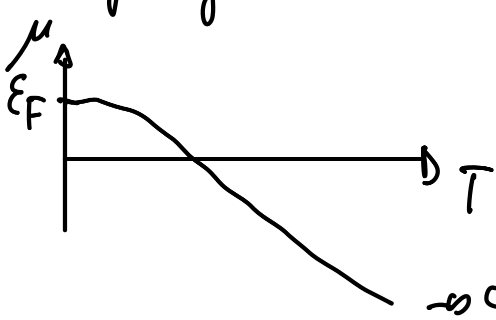


Density: $N = \sum_{|k|<k_F} g = g \frac{V}{(2\pi)^3} \int_{|k|<k_F} d^3k$

$$\boxed{N = g\frac{V}{6\pi^2} k_F^3}$$

$$\Rightarrow k_F = \left(\frac{6\pi^2 \rho_0}{g}\right)^{1/3} \text{ with } \rho_0 = \frac{N}{V} \quad \& \quad \boxed{\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2\rho_0}{g}\right)^{2/3}}$$

Canonical perspective everything above holds with $\mu(T)$ in the large system size limit



$\to$ classical limit

How low should we go to observe the T=0 limit?

Depends on the system :-)

Metal $\simeq$ cristal of atoms + electrons

Copper: $\rho_m = \frac{M}{V} = 9\ g/cm^3$, $M = m N_a = 63.5\ g/mol$ $\Big\}$ $\rho_0 = \frac{\rho_m}{m} = N_a \frac{\rho_m}{M} \simeq 10^{29}\ m^{-3}$

$\Rightarrow d = 10^{-10}\ m$ between atoms

$$\lambda_{Cu} = \sqrt{\frac{h^2}{2\pi m k T}} \simeq 1.3 \times 10^{-11}\ m \text{ at } 300K \Rightarrow \lambda_{Cu} \ll d$$

$\Rightarrow$ classical

$$\lambda_e = \sqrt{\frac{h^2}{2\pi m_e k T}} = 40 \times 10^{-10}\ m \gg d$$

$\Rightarrow$ quantum effects are important

$$\varepsilon_F = \frac{\hbar^2}{2m_e}\left(\frac{6\pi^2 \rho_0}{g}\right)^{2/3} = 7\ eV \quad \text{vs} \quad kT = 0{,}024\ eV \text{ at } 300\,^\circ K$$

$$T_F = \frac{\varepsilon_F}{k_B} = 10^4\ K \Rightarrow e^- \simeq \text{very low } T \text{ gas of fermions}$$

Q: How does the T=0 picture change at small but finite T?

① Thermodynamic: $G(z) \Rightarrow P(z), \langle E\rangle(z), \rho_0(z) = \frac{\langle N\rangle(z)}{V}$

② low temperature expansion to get z as a series in ρ_0, T, etc.

③ Then gets $\langle E\rangle(\rho_0, T)$, $P(\rho_0, T)$, etc.

1) Thermodynamics

$$G = -g k_B T \sum_{\vec{k}} \ln\left[1 + z e^{-\beta \frac{\hbar^2 k^2}{2m}}\right] \simeq -g k_B T \frac{V}{(2\pi)^3} 4\pi \int k^2 dk \ln\left(1 + z e^{-\beta \frac{\hbar^2 k^2}{2m}}\right)$$

$x = \frac{\hbar^2 k^2}{2 m k_B T}$; $k = \sqrt{x} \frac{\sqrt{2 m k_B T \pi}}{h} 2\sqrt{\pi}$; $dk = \frac{1}{\sqrt{x}} \frac{\sqrt{\pi}}{\Lambda^{1/2}}$

$$\Rightarrow G = -g k_B T \frac{V}{\Lambda^3} \frac{2}{\sqrt{\pi}} \underbrace{\int_0^\infty dx\, x^{1/2} \ln(1 + z e^{-x})}_{+\frac{2}{3}\int_0^\infty dx \frac{x^{3/2} z e^{-x}}{1 + z e^{-x}}} \Rightarrow \boxed{G = -g k_B T \frac{V}{\Lambda^3} f^-_{5/2}(z)}$$

$$G \text{ extensive} \Rightarrow \boxed{P = -\frac{G}{V} = g \frac{k_B T}{\Lambda^3} f^-_{5/2}(z)}$$

$$\langle N \rangle = z \partial_z \ln Q = -\beta z \partial_z G = g \frac{V}{\Lambda^3} z \partial_z f^-_{5/2}(z) \quad \text{but } z \partial_z f^-_m(z) = f^-_{m-1}(z)$$

$$\langle N \rangle = g \frac{V}{\Lambda^3} f^-_{3/2}(z) \Rightarrow \boxed{\rho_0 = \frac{g}{\Lambda^3} f^-_{3/2}(z)}$$

$$\boxed{\langle E \rangle = \partial_\beta (\beta G)\big|_z = \frac{3}{2} k_B T \frac{gV}{\Lambda^3} f^-_{5/2}(z) = \frac{3}{2} PV}$$ as for the ideal gas of bosons.

2) Low temperature expansion

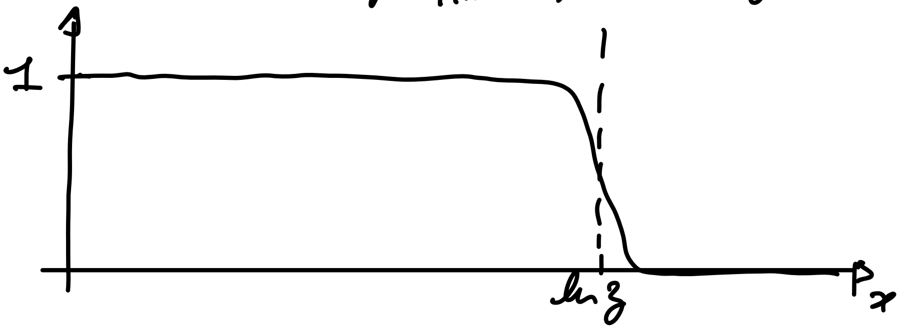
Bosons, $T \to 0$ at fixed $\rho_0 \Leftrightarrow z \to 1$. What about here?

$$\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2} \frac{\rho_0}{g} = f^-_{3/2}(z) = \frac{\sqrt{\pi}}{2} \int_0^\infty du \frac{z x^{1/2}}{e^x + z}$$

$T \to 0 \Rightarrow f^-_{3/2}(z) \to \infty \Rightarrow z$ large, consistently with $z = e^{\beta\mu}$ and ($\mu \to \varepsilon_F > 0$ + $\beta \to \infty$).

Leading order: let's try to guess the behavior of $f_m^-(z)$ at large z

$$\frac{1}{1+e^{x-\ln z}} = \begin{cases} 0 & \text{if } x \ll \ln z \\ 1 & \text{if } x \gg \ln z \end{cases}$$



step function localized at $x = \ln z \Rightarrow f_m = \frac{1}{(m-1)!}\int_0^\infty dx \, \frac{x^{m-1}}{1+e^{x-\ln z}} \simeq \frac{\int_0^{\ln z} dx\, x^{m-1}}{(m-1)!} = \frac{(\ln z)^m}{m!}$

$\Rightarrow f_m \propto (\ln z)^m$ for large z. Higher order! Sommerfeld expansion.

Since everything happens close to $x = \ln z \Rightarrow t = x - \ln z$

$$f_m^-(z) = \frac{1}{(m-1)!}\int_{-\ln z}^{+\infty} dt\, \frac{(t+\ln z)^{m-1}}{1+e^t} \xrightarrow[\text{IBP}]{} = -\frac{1}{m!}\int_{-\ln z}^{+\infty} dt\, (t+\ln z)^m \frac{e^t}{(1+e^t)^2}$$

to extract the leading order $(\ln z)^m$

↳ can be replaced by $-\infty$ since only $\neq 0$ close to $t=0$

(i) Expand $(t+\ln z)^m = (\ln z)^m \left(1+\frac{t}{\ln z}\right)^m = (\ln z)^m \sum_{k=0}^\infty \frac{m(m-1)\dots(m-k)}{k!}\frac{t^k}{(\ln z)^k}$

$m \notin \mathbb{Z}^+$ $\quad = (\ln z)^m \sum_{k=0}^\infty \frac{m!}{k!(m-k)!}\frac{t^k}{(\ln z)^k}$

(ii) $\int_{-\infty}^{+\infty} dt\, t^k \frac{e^t}{(1+e^t)^2}$ $= 0$ if k is odd by symmetry.

$= 1$ if $k = 0$

$= -2\int_0^\infty dt\, \frac{k\, t^{k-1}}{1+e^t}$ if k is even since $\int_{-\infty}^\infty = 2\int_0^\infty$ + IBP.

$k = 2p$

$$\Rightarrow f_m^-(z) = \frac{(\ln z)^m}{m!}\left(1 + \sum_{p=1}^\infty \frac{m!(\ln z)^{-2p}}{(m-2p)!} \underbrace{\frac{2}{(2p-1)!}\int_0^\infty dt\, \frac{t^{2k-1}}{1+e^t}}_{2 f_{2p}^-(1)}\right)$$

$$f_m^-(z) = \frac{(\ln z)^m}{m!}\left(1+2\sum_{p=0}^{\infty} \frac{m!\, f_{2p}^-(1)}{(m-2p)!}(\ln z)^{-2p}\right)$$

$$f_m^-(z) = \frac{(\ln z)^m}{m!}\left[1+\frac{\pi^2}{6}\frac{m(m-1)}{(\ln z)^2}+\dots\right]$$

density: $\rho_0 \simeq \frac{g}{\Lambda^3}\frac{(\ln z)^{3/2}}{3/2\,!}\left(1+\frac{\pi^2}{8}\cdot\frac{1}{(\ln z)^2}\right)$

3 Invert $z(\rho_0,T)$ to get $P(\rho_0,T)$, $E(\rho_0,T)$, etc.

$$\ln z \approx \left[\frac{3}{2}\frac{\sqrt{\pi}}{2}\frac{\Lambda^3\rho_0}{g}\right]^{2/3}\left(1+\frac{\pi^2}{8}\frac{1}{(\ln z)^2}\right)^{-2/3} \quad ; \quad \Lambda^2 = \frac{h^2}{2\pi m k_B T} = \frac{\hbar^2}{2m k_B T}4\pi$$

$$= \underbrace{\frac{\hbar^2}{2m k_B T}\left(6\pi^2\frac{\rho_0}{g}\right)}_{\frac{\varepsilon_F}{k_B T}}\left[1+\frac{\pi^2}{8}\left(\frac{k_B T}{\varepsilon_F}\right)^2\right]^{-2/3}$$

, $k_B T_F \equiv \varepsilon_F$ with T_F the Fermi temperature

$$\ln z = \beta\varepsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right] \Rightarrow \rho_0$$

Chemical potential:

$\ln z = \beta\mu \Rightarrow \mu = \varepsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right]$; μ changes sign for $T \simeq T_F$.

Pressure: $P = g\frac{kT}{\Lambda^3}\frac{(\ln z)^{5/2}}{\frac{5}{2}\,\frac{3}{2}!}\left(1+\frac{\pi^2}{6}\cdot\frac{15}{4}\cdot\frac{1}{(\ln z)^2}\right)$

$$P = \frac{2\,g\,kT}{5\cdot\frac{3}{2}!\,\Lambda^3}(\beta\varepsilon_F)^{5/2}\left(1-\frac{5\pi^2}{24}\left(\frac{T}{T_F}\right)^2+\dots\right)\left(1+\frac{5\pi^2}{8}\left(\frac{T}{T_F}\right)^2\right)$$

$$P = P_F\left[1+\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + o\left(\left(\frac{T}{T_F}\right)^2\right)\right]$$

where $P_F = \frac{2}{5}\frac{g\,kT}{\lambda^3\,\frac{3}{2}!}(\beta\varepsilon_F)^{5/2}$ is the Fermi pressure.

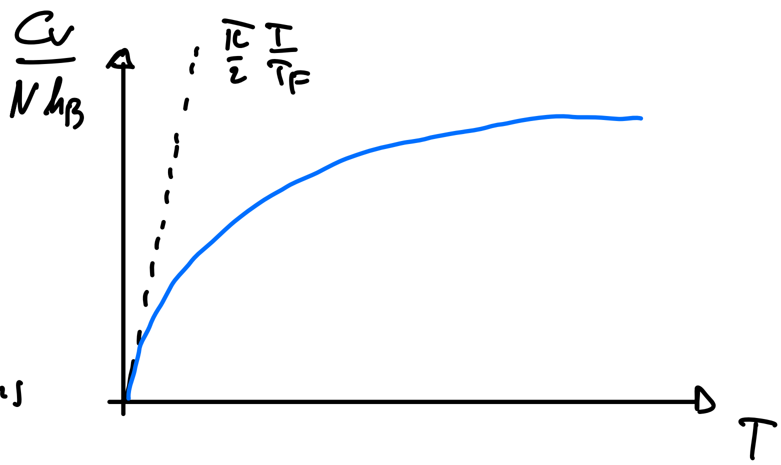
Using $\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2\rho_0}{g}\right)^{2/3}$, we find $\rho_0 = \frac{4g(\beta\varepsilon_F)^{3/2}}{3\sqrt{\pi}\,\lambda^3}$ & $\boxed{P_F = \frac{2}{5}\rho_0\varepsilon_F}$

At $T=0$, the Fermi gas has a finite pressure because excited states with non-zero momenta are occupied.

Energy: $E = \frac{3}{2}PV = \frac{3}{2}P_F V\left[1+\frac{5}{12}\pi^2\left(\frac{T}{T_F}\right)^2\right]$ with $T_F = \frac{\varepsilon_F}{k_B}$ the Fermi temperature.

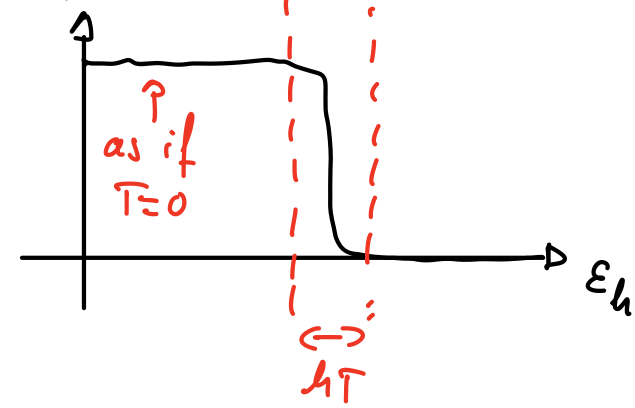
$$E = \frac{3}{5}\rho_0 V k_B T_F\left[1+\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right]$$

$$C_V = \frac{dE}{dT} = \frac{N k_B \pi^2}{2}\frac{T}{T_F}$$



$C_V \underset{T\to 0}{\propto} T$ is generic to Fermi gas in all dimensions

Intuitive explanation:



Only the fermions with $\varepsilon_k - \varepsilon_F \sim kT$ feels the temperature

An increase of temperature by δT excites a fraction $\frac{T}{T_f}$ of fermions

$$\Rightarrow \delta E = k\,\delta T \times N\frac{T}{T_f} \Rightarrow \frac{\delta E}{\delta T} \propto N k \frac{T}{T_f}$$

Consequence: For ionic solids, we had found the Debye law $C_V \propto T^3$

For metals, instead $C_V \propto T$ due to the free electrons & $T \underset{T \to 0}{\gg} T^3$

Many other applications of Fermi-Dirac statistics

* conduction in metals & solid-state physics
* liquid phase of He^3
* structure of atomic nuclei because protons & neutrons are Fermions
* structure of some stars (e.g. white Dwarf where P_{e^-} dominates)